SECTION 3.7: THE CHAIN RULE

RECALL: Given two functions, f and g, where x is in the domain of g and $g(x)$ is in the domain of f, the **composite** of f with g, denoted $f \circ g$ is defined as $(f \circ g)(x) = f(g(x))$.

Procedurally, a composite function is the result of applying the function g to an input first then applying the function f to the output from g. Schematically:

$$x \xrightarrow{\text{apply } g} g(x) \xrightarrow{\text{apply } f} f(g(x))$$

EXAMPLE 1: Find a formula for the composite function $(f \circ g)(x)$ and state the domain of $f \circ g$.

1. $f(x) = x^2$, $g(x) = 2x + 1$.

   $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2$.

   Since the domain of f and g are all real numbers, the domain of $f \circ g$ is $(-\infty, \infty)$.

2. $f(x) = \sqrt{x}$, $g(x) = 1 - x$.

   $(f \circ g)(x) = f(g(x)) = f(1-x) = \sqrt{1-x}$.

   The domain of f is $x \geq 0$ so we need $g(x) = 1 - x \geq 0$ or $x \leq 1$. Hence, the domain of $f \circ g$ is $(-\infty, 1]$

3. $f(x) = \dfrac{1}{x}$, $g(x) = 3x + 2$.

   $(f \circ g)(x) = f(g(x)) = f(3x+2) = \dfrac{1}{3x+2}$.

   The domain of f is $x \neq 0$ so we need $g(x) = 3x + 2 \neq 0$ or $x \neq -\dfrac{2}{3}$.

   Hence, the domain of $f \circ g$ is $\left(-\infty, -\dfrac{2}{3}\right) \cup \left(-\dfrac{2}{3}, \infty\right)$.

4. $f(x) = \cos(x)$, $g(x) = 3x$

   $(f \circ g)(x) = f(g(x)) = f(3x) = \cos(3x)$.

   Since the domain of f and g are all real numbers, the domain of $f \circ g$ is $(-\infty, \infty)$.

5. $f(x) = x^2$, $g(x) = \sec(x)$

   $(f \circ g)(x) = f(g(x)) = f(\sec(x)) = (\sec(x))^2 = \sec^2(x)$.

   The domain of f is all real numbers, but the domain of g is restricted so $x \neq \dfrac{(2k+1)\pi}{2}$ for integers k.

   Hence, the domain of $f \circ g$ is $\left\{ x \,|\, x \neq \dfrac{(2k+1)\pi}{2},\ k = 0, \pm 1, \pm 2, \ldots \right\}$.

EXAMPLE 2: Decompose the given function h into a composite: $h = f \circ g$. Check your answer.

HINT: Since $(f \circ g)(x) = f(g(x))$, think of g as the 'inside' function and f as the 'outside' function. Alternatively, think of f as the function you apply **last** and g as what you're applying f to.

1. $h(x) = (3x+5)^{10}$

   $g(x) = 3x+5$ and $f(x) = x^{10}$

   Check: $(f \circ g)(x) = f(g(x)) = f(3x+5) = (3x+5)^{10}$ ✓

2. $h(x) = \sqrt[3]{1-3x}$

   $g(x) = 1-3x$ and $f(x) = \sqrt[3]{x}$

   Check: $(f \circ g)(x) = f(g(x)) = f(1-3x) = \sqrt[3]{1-3x}$ ✓

3. $h(x) = \dfrac{1}{x^2+1}$

   $g(x) = x^2+1$ and $f(x) = \dfrac{1}{x}$

   Check: $(f \circ g)(x) = f(g(x)) = f\left(x^2+1\right) = \dfrac{1}{x^2+1}$ ✓

4. $h(\theta) = \tan(5\theta)$

   $g(\theta) = 5\theta$ and $f(\theta) = \tan(\theta)$

   Check: $(f \circ g)(\theta) = f(g(\theta)) = f(5\theta) = \tan(5\theta)$ ✓

5. $h(t) = \sin^3(t) = (\sin(t))^3$

   $g(t) = \sin(t)$ and $f(t) = t^3$

   Check: $(f \circ g)(t) = f(g(t)) = f(\sin(t)) = (\sin(t))^3 = \sin^3(t)$ ✓

EXAMPLE 3: Decompose $F(\theta) = \sec^4(2\theta)$ into a composite of three functions: $F = f \circ g \circ h$.

Since $(f \circ g \circ h)(\theta) = f(g(h(\theta)))$, we write $F(\theta) = \sec^4(2\theta) = (\sec(2\theta))^4$ and working from the inside out:

$h(\theta) = 2\theta$, $g(\theta) = \sec(\theta)$ and $f(\theta) = \theta^4$.

Check: $(f \circ g \circ h)(\theta) = f(g(h(\theta))) = f(g(2\theta)) = f(\sec(2\theta)) = (\sec(2\theta))^4 = \sec^4(2\theta)$ ✓

We're now ready to see how derivatives interact with composite functions.

CHAIN RULE: $D_x[f(g(x))] = f'(g(x))g'(x)$. Said differently, if we let $u = g(x)$, then:

$$D_x[f(u)] = f'(u) \cdot u'$$

NOTE: The function u is often called the 'inside' function whilst the function f is called the 'outside' function.

EXAMPLE 4: Find the indicated derivative.

1. For $h(x) = (3x+5)^{10}$, find $h'(x)$.

   The 'inside' function is $u = 3x+5$, the 'outside' function is $f(x) = x^{10}$, so $h(x) = f(u) = u^{10}$.

   Hence, $h'(x) = 10u^9\, u' = 10(3x+5)^9(3) = 30(3x+5)^9$

   In other words,

   $$D_x\left[(3x+5)^{10}\right] = 10(3x+5)^9 D_x[3x+5] = 10(3x+5)^9(3) = 30(3x+5)^9.$$

2. $h(x) = \sqrt[3]{1-3x} = (1-3x)^{1/3}$

   The 'inside' function is $u = 1-3x$, the 'outside' function is $f(x) = x^{1/3}$, so $h(x) = f(u) = u^{1/3}$.

   Hence, $h'(x) = \frac{1}{3}u^{-2/3}\, u' = \frac{1}{3}(1-3x)^{-2/3}(-3) = -(1-3x)^{-2/3}$

   In other words,

   $$D_x\left[\sqrt[3]{1-3x}\right] = D_x\left[(1-3x)^{1/3}\right] = \frac{1}{3}(1-3x)^{-2/3}D_x[1-3x] = \frac{1}{3}(1-3x)^{-2/3}(-3) = -(1-3x)^{-2/3}.$$

3. $h(x) = \dfrac{1}{x^2+1} = \left(x^2+1\right)^{-1}$

   The 'inside' function is $u = x^2+1$, the 'outside' function is $f(x) = x^{-1}$, so $h(x) = f(u) = u^{-1}$.

   Hence, $h'(x) = (-1)u^{-2}\, u' = -\left(x^2+1\right)^{-2}(2x) = -\dfrac{2x}{(x^2+1)^2}$

   In other words,

   $$D_x\left[\frac{1}{x^2+1}\right] = D_x\left[(x^2+1)^{-1}\right] = (-1)(x^2+1)^{-2}D_x\left[x^2+1\right] = (-1)(x^2+1)^{-2}(2x) = -\frac{2x}{(x^2+1)^2}.$$

4. $h(\theta) = \tan(5\theta)$

   The 'inside' function is $u = 5\theta$, the 'outside' function is $f(\theta) = \tan(\theta)$, so $h(\theta) = f(u) = \tan(u)$.

   Hence, $h'(\theta) = \sec^2(u)\, u' = \sec^2(5\theta)(5) = 5\sec^2(5\theta)$.

   In other words,

   $$D_\theta\left[\tan(5\theta)\right] = \sec^2(5\theta)D_\theta[5\theta] = \sec^2(5\theta)(5) = 5\sec^2(5\theta).$$

5. $h(t) = \sin^3(t) = (\sin(t))^3$

   The 'inside' function is $u = \sin(t)$, the 'outside' function is $f(t) = t^3$, so $h(t) = f(u) = u^3$.

   Hence, $h'(t) = 3u^2\, u' = 3(\sin(t))^2\cos(t) = 3\sin^2(t)\cos(t)$.

   In other words,

   $$D_t\left[\sin^3(t)\right] = D_t\left[(\sin(t))^3\right] = 3(\sin(t))^2 D_t[\sin(t)] = 3(\sin(t))^2\cos(t) = 3\sin^2(t)\cos(t).$$

We combine the chain rule with the power rule and the derivative rules for the circular functions to get:

GENERALIZED DERIVATIVE RULES: If u is a differentiable function of x, then:

- **POWER RULE:** $D_x[u^k] = ku^{k-1} \cdot u'$
- $D_x[\sin(u)] = \cos(u) \cdot u'$
- $D_x[\cos(u)] = -\sin(u) \cdot u'$
- $D_x[\sec(u)] = \sec(u)\tan(u) \cdot u'$
- $D_x[\csc(u)] = -\csc(u)\cot(u) \cdot u'$
- $D_x[\tan(u)] = \sec^2(u) \cdot u'$
- $D_x[\cot(u)] = -\csc^2(u) \cdot u'$

EXAMPLE 5: (VIDEO) Find the indicated derivative.

1. For $r(x) = \dfrac{5}{\sqrt{2x-7}}$, find $r'(x)$.

Ans: $r'(x) = -5(2x-7)^{-3/2}$

2. For $y = \cos^5(\theta)$, find $\dfrac{dy}{d\theta}$.

Ans: $\dfrac{dy}{d\theta} = -5\cos^4(\theta)\sin(\theta)$

3. $D_x\left[x^2\sin(3x)\right]$.

Ans: $D_x\left[x^2\sin(3x)\right] = 2x\sin(3x) + 3x^2\cos(3x)$

EXAMPLE 6: (VIDEO) Find the indicated derivative.

1. Find $f'(t)$ for $f(t) = \dfrac{\cos(2t)}{1+\sin(2t)}$.

Ans: $f'(t) = \dfrac{-2}{1+\sin(2t)}$

2. Find y' for $y = \sqrt[3]{\dfrac{2x-3}{2x+3}}$.

Ans: $y' = \dfrac{4}{(2x-3)^{2/3}(2x+3)^{4/3}}$

3. Find $D_t^2[\sec(3t)]$.

Ans: $D_t^2[\sec(3t)] = 9\sec(3t)\tan^2(3t) + 9\sec^3(3t)$

THE CHAIN RULE IN LEIBNIZ NOTATION

Using Leibniz notation, we can view the chain rule as: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

More specifically, if $x = a$ and $u(a) = b$, then: $\left.\dfrac{dy}{dx}\right|_{x=a} = \left.\dfrac{dy}{du}\right|_{u=b} \cdot \left.\dfrac{du}{dx}\right|_{x=a}$.

In words this equation says: the rate of change of y with respect to x when $x = a$ is the product of the rate of change of y with respect to u when $u = b$ times the rate of change of u with respect to x when $x = a$.

EXAMPLE 7: Suppose $x(p)$ gives the number of 'doubries' sold, x, as a function of the price per doubrie p, in dollars. Let $P(x)$ denote the profit, in dollars, obtained by selling x doubries.

1. Interpret the equation: $x(5) = 20$ in terms of dollars and doubries.

   $x(5) = 20$ means that when the price is set to \$5 per doubrie, we will sell 20 doubries.

2. Interpret the equation: $x'(5) = -2$ in terms of dollars and doubries.

   This means when the price is \$5 per doubrie, increasing the price by \$1 results in selling 2 fewer doubries.

3. Interpret the equation $P(20) = 500$ in terms of dollars and doubries.

   This means when 20 doubries are sold, the profit is \$500.

4. Interpret the equation $P'(20) = -30$ in terms of dollars and doubries.

   This means when 20 doubries are sold, for each additional doubrie sold, the profit will decrease by \$30.

5. Use the chain rule to find the rate of change of profit with respect to price when the price is \$5 per doubrie.

   HINT: Leibniz Notation works best here: $\dfrac{dP}{dp} = \dfrac{dP}{dx} \cdot \dfrac{dx}{dp}$

   We know when $p = 5$, $x = 20$. Hence to find $\dfrac{dP}{dp}$ we when $p = 5$, use the following:

   $P'(20) = \dfrac{dP}{dx} = -30$ and $x'(5) = \dfrac{dx}{dp} = -2$.

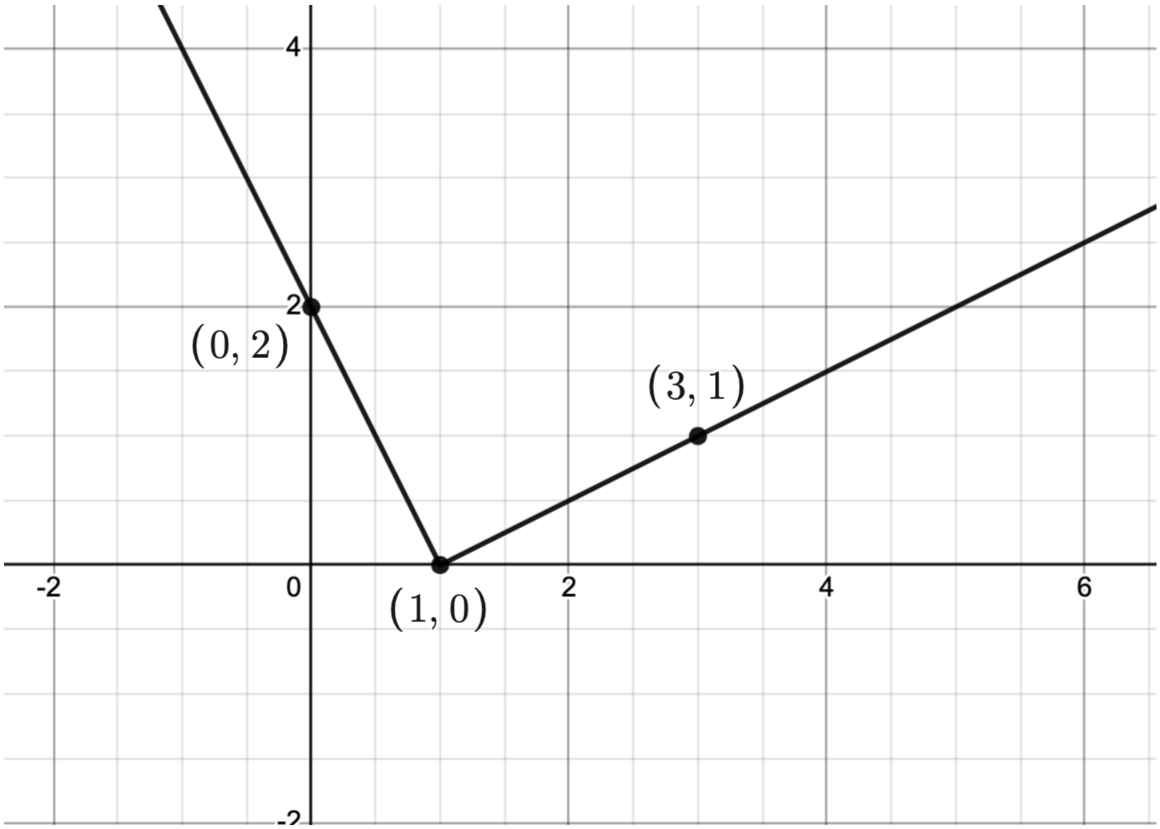
   Putting this together, when $p = 5$: $\left.\dfrac{dP}{dp}\right|_{p=5} = \left.\dfrac{dP}{dx}\right|_{x=20} \cdot \left.\dfrac{dx}{dp}\right|_{p=5} = (-30)(-2) = 60$.

6. If the price is currently \$5 per doubrie, do we raise or lower the price to increase profit? Explain.

   Since $\dfrac{dP}{dp} = 60$ when $p = 5$, for each additional \$1 increase in price, the profit should increase by \$60.

   Hence, to increase profit, we increase the price. While this seems counter-intuitive, consider that increasing the price usually means lower demand, and hence, lower manufacturing costs. (Provided we produce as many doubries as we sell . . .)

EXAMPLE 8: (VIDEO) Suppose $f(x) = 3x - x^2$ and the graph of $y = g(x)$ is given below.



$y = g(x)$

Find the indicated values.

1. If $h(x) = (f \circ g)(x)$, find $h'(0)$ and $h'(3)$.

2. If $j(x) = (g \circ f)(x)$, find $j'(0)$ and $j'(3)$.

HOMEWORK: Section 3.7: 1 - 71 odd, 75, 77, 89*, 91*

APPENDIX: PROOF OF THE CHAIN RULE

To prove the chain rule, we use local linearity. Suppose g is differentiable at $x = a$ and f is differentiable at $g(a)$.

Since g is differentiable at $x = a$, we know: $g'(a) = \lim_{x \to a} \dfrac{g(x) - g(a)}{x - a}$.

Since f is differentiable at $g(a)$, we use the local linearity theorem for f, with $g(x)$ for x and $g(a)$ for a:

$$f(g(x)) = f'(g(a))[g(x) - g(a)] + f(g(a)) + \epsilon(g(x))[g(x) - g(a)],$$

for some function $\epsilon(g(x))$ such that $\epsilon(g(x)) \to 0$ as $g(x) \to g(a)$.

Moreover, since g is differentiable at $x = a$, g is continuous at $x = a$. This means that as $x \to a$, $g(x) \to g(a)$ so:

$$\lim_{x \to a} \epsilon(g(x)) = 0.$$

Using local linearity, the difference quotient:

$$\begin{aligned}
\frac{f(g(x)) - f(g(a))}{x - a} &= \frac{f'(g(a))[g(x) - g(a)] + f(g(a)) + \epsilon(g(x))[g(x) - g(a)] - f(g(a))}{x - a} \\
&= f'(g(a))\,\frac{g(x) - g(a)}{x - a} + \epsilon(g(x))\,\frac{g(x) - g(a)}{x - a}
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} &= \lim_{x \to a} \left[f'(g(a))\,\frac{g(x) - g(a)}{x - a} + \epsilon(g(x))\,\frac{g(x) - g(a)}{x - a} \right] \\
&= \lim_{x \to a} \left[f'(g(a))\,\frac{g(x) - g(a)}{x - a} \right] + \lim_{x \to a} \left[\epsilon(g(x))\,\frac{g(x) - g(a)}{x - a} \right] \\
&= f'(g(a)) \lim_{x \to a} \left[\frac{g(x) - g(a)}{x - a} \right] + \left[\lim_{x \to a} \epsilon(g(x)) \right] \left[\lim_{x \to a} \frac{g(x) - g(a)}{x - a} \right] \\
&= f'(g(a))g'(a) + 0 \cdot g'(a) = f'(g(a))g'(a) \checkmark
\end{aligned}$$

It follows that $D_x[f(g(x))] = f'(g(x))g'(x)$.